Capital Asset Pricing Model (CAPM)

- **Assumptions**

  i) Investors are risk-averse.

  ii) Investors price-takers and have homogeneous expectations. → The returns distribution is jointly normal (multivariate normal).

  iii) Investors borrow and lend at R_f rate.

  iv) Quantity of assets is fixed. All assets are divisible and marketable.

  v) Market is frictionless and information is costless.

  vi) Market is perfect → No taxes, no transaction cost and no restrictions on short-sell.

Let $a = w_i \quad and \quad 1 - a = w_m$

$$E[\tilde{R}_P] = aE[\tilde{R}_i] + (1-a)E[\tilde{R}_m] \quad \rightarrow \quad \frac{\partial E[\tilde{R}_P]}{\partial a} = E[\tilde{R}_i] - E[\tilde{R}_m]$$

$$\sigma_P = [a^2\sigma_i^2 + (1-a)^2\sigma_m^2 + 2a(1-a)\sigma_{im}]^{\frac{1}{2}} \rightarrow \frac{\partial \sigma_P}{\partial a} = \frac{1}{2}[a^2\sigma_i^2 + (1-a)^2\sigma_m^2 + 2a(1-a)\sigma_{im}]^{-\frac{1}{2}}$$

$$\times [2a\sigma_i^2 - 2\sigma_m^2 + 2a\sigma_m^2 + 2\sigma_{im} - 4a\sigma_{im}] = \frac{\sigma_{im} - \sigma_m^2}{\sigma_m}$$

$$\left.\frac{\partial E[\tilde{R}_P]/\partial a}{\partial \sigma_P / \partial a}\right|_{ai=0} = \frac{E[\tilde{R}_i] - E[\tilde{R}_m]}{(\sigma_{im} - \sigma_m^2)/\sigma_m} = slope\ of\ K - market\ line = \frac{E[R_m] - R_f}{\sigma_m}$$

(Because at equilibrium "M" there can be no excess demand → a = 0.)

$$\frac{E[\tilde{R}_i] - E[\tilde{R}_m]}{(\sigma_{im} - \sigma_m^2)/\sigma_m} = \frac{E[R_m] - R_f}{\sigma_m} \rightarrow E[\tilde{R}_i] = R_f + [E[R_m] - R_f]\frac{\sigma_{im}}{\sigma_m^2}$$

Required ROR = R_f + Risk Premium×β_i

<u>Less Risky ← $\beta_i < \beta_m = 1$</u> < β_I → Riskier and More Volatile

Co-moves with market

Properties of CAPM

1. In equilibrium all assets must be priced such that they fall exactly on Security Market Line ($\beta_m = 1$).

$$\tilde{R}_i = a_i + b_i R_m + \varepsilon_i \quad \leftarrow \quad \textit{historical return}$$

$$\sigma_i^2 = b_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2, \textit{ where } b_i^2 \textit{ is weight}, \sigma_m^2 \textit{ is systematic risk and } \sigma_{\varepsilon i}^2 \textit{ is nonsystematic risk.}$$

2. The β for the portfolio is the weighted sum of the β's for the assets: $\beta_P = \sum_{i=1}^{N} w_i \beta_i$

3. P_t and CAPM (Risk-Adjusted ROR Valuation Formula)

$$\frac{E[P_{t+1}] - P_t}{P_t} = E[\tilde{R}_{it+1}] = k_S = R_f + (E[R_{mt}] - R_f)\beta_i, \text{ where } k_S = \text{cost of equity capital}$$

$$P_t = \frac{E[\tilde{P}_{t+1}]}{1 + R_f + (E[R_{mt}] - R_f)\beta_i}$$

$$\sigma_t^2[\tilde{R}_i] = \sigma_{it}^2 = E\left[\frac{\tilde{P}_{t+1} - P_t}{P_t} - E\left[\frac{\tilde{P}_{t+1} - P_t}{P_t}\right]\right]^2 = \frac{1}{P_t^2}\sigma_t^2[\tilde{P}_{t+1}]$$

$$\sigma_t[\tilde{R}_i \tilde{R}_m] = E\left[\frac{\tilde{P}_{t+1} - P_t}{P_t} - E\left[\frac{\tilde{P}_{t+1} - P_t}{P_t}\right]\right]\left[\tilde{R}_m - E[\tilde{R}_m]\right] = \frac{1}{P_t}\sigma_t[P_{t+1}\tilde{R}_m]$$

$$\beta_i = \frac{\sigma_t[\tilde{R}_i \tilde{R}_m]}{\sigma_{mt}^2} = \frac{1}{P_t} \times \frac{\sigma_t[\tilde{P}_{t+1}, P_t]}{\sigma_{mt}^2}$$

$$E[Project] = R_f + (E[R_m] - R_f)\beta_{Project} \geq k_{Project}$$

Extension of CAPM

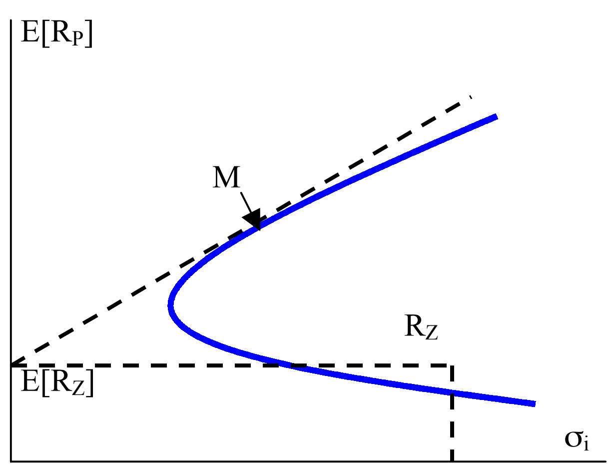
1. No R_f Asset

$$E[\tilde{R}_P] = w_i E[R_m] + (1 - w_i)E[\tilde{R}_Z]$$

$$E[\tilde{R}_i] = E[\tilde{R}_Z] + (E[\tilde{R}_m] - E[\tilde{R}_Z])\beta_i$$

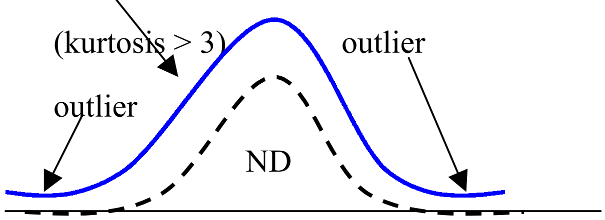
where $E[R_Z]$ is a portfolio having zero covariance with R_m.

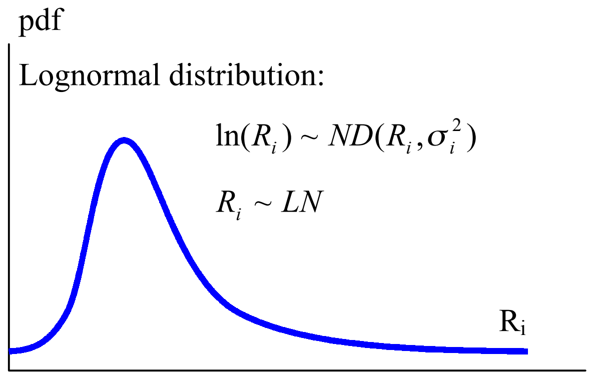
Even without R_f asset, $E[R_Z]$ works effectively as R_f portfolio.



2. Non-Normal Distribution of Returns

Actual return may be likely to be fat-tailed.





3. Non-marketable Assets (Human K)

$E[\tilde{R}_i] = Rf + (E[\tilde{R}_m] - R_f)\left(\frac{V_m \sigma_{im} + \sigma_{iH}}{V_m \sigma_m^2 + \sigma_{mH}}\right)$, where V_m is the value of marketable securities, and H is the non-marketable asset.

4. Multiperiod Model (As long as R_f is constant)

$E[R_{it}] = R_f + (E[R_{mt}] - R_f)\beta_{it} = R_f + \gamma_1(E[R_m] - R_f) + \gamma_2(E[R_N] - R_f)$, where R_N is the return on assets negatively correlated with R_f. (?)

5. Heterogeneous Expectations

   Model invalid.

Testing CAPM

1. Historical (Empirical) Returns in Observed Form

$R_{it} = E[R_{it}] + (R_{mt} - E[R_{mt}])\beta_i + \varepsilon_{it}$, where

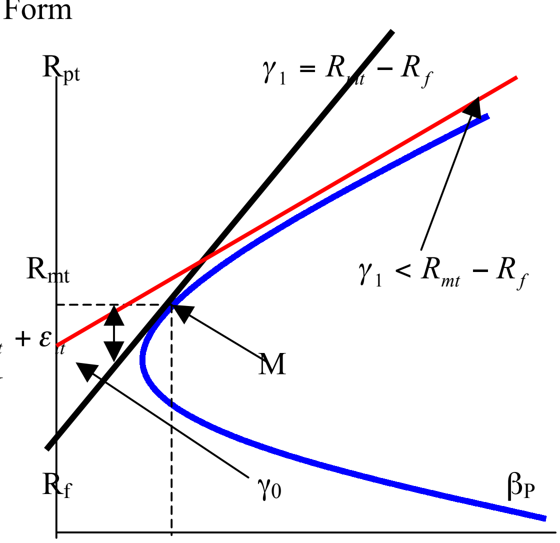
$E[R_{mt} - E[R_{mt}]] = 0 \rightarrow$ Fair Game

$E[\varepsilon_{it}] = 0 \rightarrow$ White Noise

Take expectations on both sides.

$$R_{it} = R_f + (E[R_{mt}] - R_f)\beta_{it} + (R_{mt} - E[R_{mt}])\beta_{it} + \varepsilon_{it}$$
$$= R_f + (R_{mt} - R_f)\beta_{it} + \varepsilon_{it} \leftarrow \textit{ex post CAPM}$$

$$R_{Pt} - R_f = \gamma_0 + (R_{mt} - R_f)\beta_P + \varepsilon_{Pt}$$
$$R_{Pt}' = \gamma_0 + \gamma_1\beta_P + \varepsilon_{Pt} \leftarrow \textit{emprical form}$$



Hypothesis	Results
$\gamma_0 = 0$	$\gamma_0 \neq 0$
$\gamma_1 = R_m - R_f$	$\gamma_1 < R_m - R_f$
The relationship is linear in β.	Low β securities earn more than CAPM predicts, and high β securities, less than predicted. β^2 (unsystematic risk) explains only a small # of time periods sampled. → β is a dominant measure of risk.
$R_m - R_f > 0$ in the LR.	$R_m - R_f > 0$ in the LR.

2. Factors other than β:

i) P/E : Low P/E returns higher than CAPM predicts.

ii) Dividend yield: High Div/Equity (D/P) requires higher return. (However, P/E & D/P ratio should be derived into the asset-pricing model theoretically, not just added in arbitrarily by empirical significance.)

iii) Firm size: Smaller firms return high abnormal returns.

iv) Seasonality: January effect

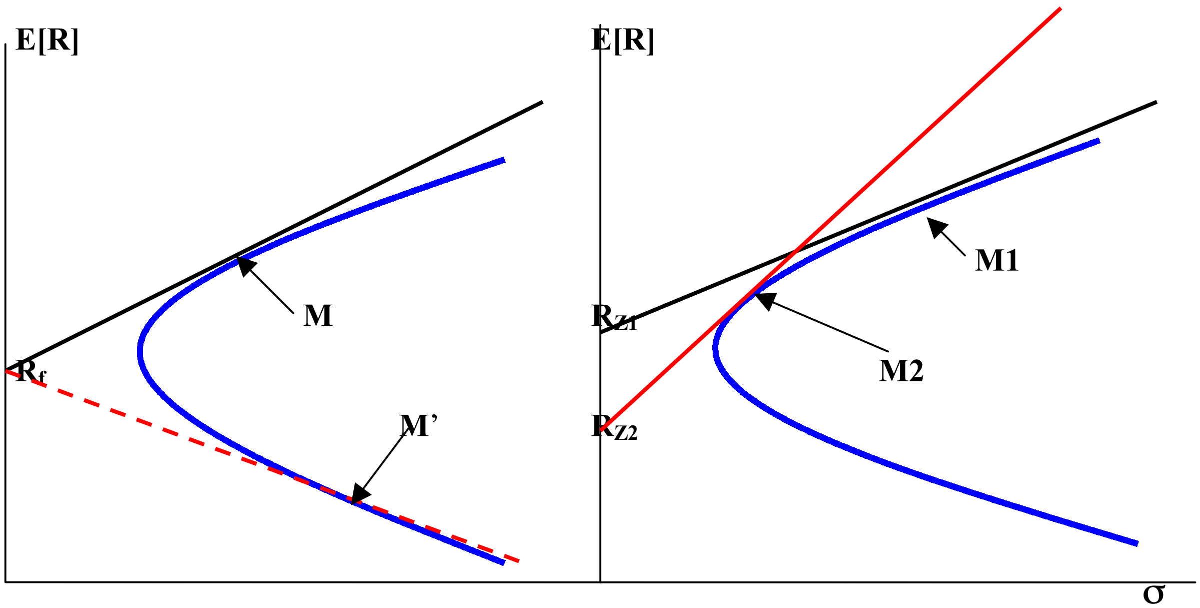
Roll's Critique

1. The only legitimate test of CAPM is whether or not 'M' is mean-variance efficient.
2. If performance is measured to an *ex post* efficient index, no security will have abnormal returns. If measured to an *ex post* efficient index, any ranking of performance is impossible.
3. Then, the cross-section security market line cannot be used to measure the *ex-post* performance.

$$* \ E[R_i] = E[R_{ZI}]+(E[R_I]-E[R_{ZI}])\beta_{iI}$$

- Nothing unique about 'M'. We can always choose any efficient portfolio as an index, then find MVP uncorrelated with efficient portfolio.
- Because the $E[R_i]$ can be written as a linear function of its β measured relative to any efficient index, it is not necessary to know the market index 'M'.

- The only way to test CAPM directly is to see whether or not the true market portfolio is *ex post* efficient, but because market portfolio contains all the assets, it is impossible to observe.



<u>Future Prices & CAPM</u>

$E[S_T] = S_0 e^{ks} \rightarrow S_0 = E[S_T]e^{-ks}$

i) ${}_0F_T = E[S_T]e^{(ks-rf)T}$, where $ks-rf$ ($ks>rf$) is the risk premium if the asset is risky.

${}_0F_T = E[S_T] = e^{(ks-rf)T} < E[S_T]$

ii) ${}_0F_T = E[S_T]$ if asset is riskless: $ks=rf$.